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# Coexistence of Observables

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Recent results in the theory of integration of complex-valued functions with respect to a positive operator-valued measure are used to generalize the usual notion of coexistent observables. This leads to a connection between effects as observables and the quantization scheme of stochastic quantum mechanics. It also leads to a new viewpoint for the concept of a "classical apparatus" for quantum measurement which does not require a classical mechanical treatment of the apparatus from the outset.

## 1. INTRODUCTION

I address the question, "How should one mathematically describe an observable in the quantum setting?" Historically (discrete) observables in the quantum measurement process of von Neumann (1955) arise from the repeatability axiom and are described by projection-valued measures (spectral measures) via the spectral theorem. More recently, however, it was seen that for continuous observables (position and momentum, for example) there were no repeatable measurements (Ozawa, 1984) from which to derive the projection operators, and that localization operators were not adequately described by projection-valued measures, but rather by positive operator-valued measures (POVs) (Jauch *et al.*, 1967). This same POV structure was determined by Ludwig (1985) in his general analysis of measurement, and is denoted "effect-valued measure" in his terminology. Furthermore, these POVs were seen to occur in the analysis of measurement of momentum and position (Holevo, 1982; Schroeck, 1981) and of spin (Schroeck, 1982; Busch, 1985*b*), allowed for the localization of the photon (Ali and Emch, 1974), allowed for the general localization of particles in the representations of the Galilei and Poincaré groups on wave functions defined over phase space (Ali and Prugovečki, 1986; Ali *et al.*, 1988; Brooke and Schroeck, to appear),

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play a central role in the analysis of quantum statistical analysis (Holevo, 1982) and quantum entropy (Schroeck, 1985a), and are presently finding application in a wide variety of practical experimental situations.

In the analysis of observables, it has been argued that linear combinations of observables are again observables (Emch, 1972). This should apply as well to these positive operators determined by an apparatus; I will review and extend that discussion below. I shall show that for a "classical" screen, with a very weak requirement for classicality, these linear combinations may be refined to give a form of integral. On the other hand, the integrals so obtained will be shown to correspond precisely with the integrals occurring in the quantization of classical observables in the stochastic quantization scheme (Schroeck, 1985*c*). Since this quantization scheme also solves the so-called ordering problem of quantization, I will have strengthened the view that the measuring apparatus itself "solves" the ordering problem. Furthermore, the set of general observables obtained via these linear combinations and integrals will all be manifestly coexistent, having been measured simultaneously by a single apparatus. Since this set is not by any means necessarily commutative, one again recovers the known result that when using POVs to describe measurements, there is no necessity for coexistent observables to commute (Lahti, 1985; Schroeck, 1985*b*).

The observables I obtain may be called "unsharp observables" (Busch, 1985*a*) because of the method of observation, and because no wavefunction can pass through the system without distortion or attenuation or both. Nonetheless, I show here that many of these observables possess purely discrete spectrum. This result, as well as the general mathematical apparatus I shall use, is based on recent analysis of integration of functions with respect to a positive operator-valued measure (Schroeck, 1988).

## 2. THE OBSERVABLES OF A SINGLE APPARATUS FORM A LINEAR SPACE

Consider an experiment for which the outcome is a record on a screen, or a track or tracks in a gel, cloud chamber, spark chamber, etc. We may partition that screen, gel, cloud chamber, or spark chamber when we attempt to "read" the results, and lump the individual events in each cell of the partition into a single number, as we do with a multichannel analyzer. If we let $X$ denote the entire space of outcomes (that is, the points on the screen, etc.), and let $\{\Delta_i\}$ be a finite partition of this space, then the process in which an incoming state with density $\rho$ gives a result in $\Delta_i$ might be described by

$$\rho \to E(\Delta_i)^{1/2} \rho E(\Delta_i)^{1/2} \tag{1}$$

where $E(\Delta_i)$ is an effect in Ludwig's sense; i.e., $E(\Delta_i)$ is a positive self-adjoint operator with

$$0 \leq E(\Delta_i) \leq 1 \tag{2a}$$

$$\sum_{i \in I} E(\Delta_i) = E\left(\bigcup_{i \in I} \Delta_i\right) \tag{2b}$$

where $\{\Delta_i | i \in I\}$ is a countable disjoint family, and

$$E(X) = 1 \tag{2c}$$

In the special case that $E(\Delta_i)$ is a projection for each $i$, then (1) is the historical collapse scheme. Furthermore, according to the Davies and Lewis (1970; Davies, 1970) measurement scheme, the observable associated with this apparatus but in which only the events in cell $\Delta_i$ are counted, is given by $E(\Delta_i)$ itself:

$$\mathrm{Tr}[E(\Delta_i)^{1/2} \rho E(\Delta_i)^{1/2}] = \mathrm{Tr}[\rho E(\Delta_i)] \tag{3}$$

Of course we may also decide to lump together the results of two cells, or any finite number of cells for that matter, requiring the expression (2b) to hold. Similarly if we decide to further partition some of the cells. On the other hand, we may have decided that some of the cells of the screen, etc., are less efficient than others, and we may wish to compensate for this by weighting the results of cell $\Delta_i$ with compensation factor $c_i \geq 0$ to obtain the expected values

$$\sum_i c_i \,\mathrm{Tr}[\rho E(\Delta_i)] = \mathrm{Tr}\left[\rho \sum_i c_i E(\Delta_i)\right] \tag{4}$$

There may be many other good experimental reasons (Emch, 1972) to consider as observables expressions of the form $\sum_i c_i E(\Delta_i)$; so, I will not restrict the $c_i$ to being real and nonnegative in what follows. For full generality I will allow the $c_i$ to be complex.

In the preceding paragraph I focused on the behavior of the observables under the process of "lumping together" the results of two cells. It would not necessarily be appropriate to focus instead on the states by adding two terms as in (1) to describe the effect of the "lumping together" on the resulting states. The process of adding terms of the form (1) creates a mixture of states and destroys correlations between them; however, the process of, say, opening two windows in a screen certainly yields correlations between the two components of the state, as the two-slit experiment demonstrates. This is exacerbated by the fact that there are infinitely many alternatives to (1) to describe the measurement and which yield the same observables. For example,

$$\rho \to U_i E(\Delta_i)^{1/2} \rho E(\Delta_i)^{1/2} U_i^\dagger \tag{1'}$$

where $U_i$ is any unitary operator, yields the same corresponding observable $E(\Delta_i)$ via (3). Adding two terms such as (1′) for any $U_i$ yields the same "lumped together" observables as adding the corresponding terms of the form (1), although the states are different. I leave the form of the state after measurement to an analysis of the object–apparatus interaction (which leads to the POV structure for the observables in any event).

Let us assume, for an example, that our screen is in fact a photographic plate for a color photograph. Then the partition $\{\Delta_i\}$ may include a separation of various parts of the color spectrum, and hence $\{\Delta_i\}$ is a partition of configuration space × color spectrum space; that is, $\{\Delta_i\}$ partitions the equivalent to phase space. More generally, any screen may have efficiency varying not only over configuration space, but also over momentum/frequency space. There may even be "windows" in momentum/frequency in which the screen is effectively transparent. Furthermore, we may have other dependences, such as charge, mass, . . . , all of which may go into the description of the screen, for more sophisticated screens. Henceforth, the outcome space $X$ will be taken to be a subset of $\mathbb{R}^n$ (or even an $n$-dimensional manifold), and need not be viewed simply as configuration space for physical interpretation. Less complicated observables may be obtained by "marginality," that is, by summing over the full range of unwanted parameters.

For practical physical interpretation and implementation, the partitions $\{\Delta_i\}_{i\in I}$ must be finite. In the case of a photographic plate, the fineness of the partition is limited by the size of the light-sensitive grains. Also, in analyzing such photographs (say, of particle tracks) where one magnifies the photograph and then converts events on the track to numerical form on a computer, one has an effective grid which provides the screen with a partition which is finite. One may refine these partitions by subdividing grid cells (until some limit such as grain size prohibits further refinement); however, as in the case with magnifying particle tracks, I will entertain the possibility of achieving the continuum limit for all practical purposes.

The discussion so far has been based on an analysis of screens. This should not be thought of as a special situation, since a similar analysis applies in a wide range of generality. I mention, for another typical example, that it also applies to signal analysis. It is well known that the process of chopping a signal (and then analyzing the pieces) does drastic things to the signal, yet is a common processing mode for much of electronics. A more sophisticated method of analysis, free of any significant band pass or time chopping of the signal, is made by comparing the signals with a standardized test signal $\tau$ in the following fashion.

Let $U(t, d)$ denote the operation on test signal $\tau$ of time shifting by $t$ and frequency shifting by $\delta$. Let $s$ be the signal to be analyzed. Suppose one superimposes these to obtain $s+U(t,\delta)\tau$ and records the intensity

$$I(t,\delta)=\int |s+U(t,\delta)\tau|^2=\int |s|^2+\int |\tau|^2+2\,\mathrm{Re}\int s^*U(t,\delta)\tau$$

where the integral is over a time interval sufficiently long to accumulate essentially all of the mixed signal. By cloning $s$ and recording $I(t,\delta)$ for a variety of choices $(t,\delta)$, one obtains enough information to isolate $|\int s^*U(t,\delta)\tau|$ by calculating the contrast (visibility) in $I(t,\delta)$ where $\tau$ or $s$ is farily monochromatic (Brooke and Schroeck to appear). However, writing

$$\left|\int s^*U(t,\delta)\tau\right|^2=\langle s, U(t,\delta)\tau\rangle\langle U(\tau,\delta)\tau, s\rangle=\langle s, a(t,\delta)s\rangle \tag{5a}$$

where

$$a(t,\delta)=U(t,\delta)|\tau\rangle\langle\tau|U(t,\delta)^\dagger \tag{5b}$$

we obtain directly from measurement the effects given by

$$E(\Delta)=\int_\Delta a(t,\delta)\,dt\,d\delta \tag{6}$$

This presumes we may generate a continuum of $(t,\delta)$ shifts with our apparatus. Practically speaking, we can only approximate this, usually with a very fine but wide range of $(t,\delta)$ shifts, approximating (6) with the analog of a Riemann sum.

I emphasize that even though the $a(t,\delta)$ are each projections, they are not orthogonal for different $(t,\delta)$, and cannot be used to analyze $s$ in the historical manner. Nonetheless, these $E(\Delta)$ are informationally complete (Klauder, 1964) in the sense that knowledge of all $\mathrm{Tr}[\rho E(\Delta)]$ for all $\Delta$, or equivalently, knowledge of all $\mathrm{Tr}[\rho a(t,\delta)]$ for all $(t,\delta)$, uniquely determines $\rho$. For only a subset of such results, $\rho$ may still be determined with high accuracy. In fact, this processing method is analogous to holography, since it encodes the full phase space (time-frequency) behavior of the signal $s$, rather than only recording its frequency behavior. I should remind the reader that complete knowledge of $|s(t)|^2$ and of $|\tilde{s}(\nu)|^2$, $\tilde{s}$ the Fourier transform of $s$, does not uniquely determine $s$ in general, just as pure position and pure momentum knowledge of $|\psi(x)|^2$, resp. $|\tilde{\psi}(k)|^2$ does not uniquely determine $\psi$ in general (Vogt, 1978; Corbett and Hurst, 1978; Prugovečki, 1984; Pavčič, 1985; Busch and Lahti, 1989). More information is stored in the joint probability distributions than in the separate "marginal" distributions; i.e., more information is accessible by means of the POV than from the union of the spectral families for momentum and for position.

I should note there that one usually does not generate $U(t, \delta)\tau$ exactly with any realistic apparatus, because of instabilities of the apparatus as well as a form of the uncertainty principle for signal processing. Thus, (6) is closer to the experimental capability than (5b). Even closer to the actual experimental situation is the form

$$A(f) \equiv \int f(t, \delta) a(t, \delta)\, dt\, d\delta \tag{7}$$

where $f$ is a probability distribution determined by the (instability of the) apparatus itself. It is also "an integral with respect to a POV"; and may be approximated, as I shall show, by $\sum_i f(t_i, \delta_i) E(\Delta_i)$, where $\Delta_i$ contains $(t_i, \delta_i)$, and $\{\Delta_i\}$ is a partition of the time-frequency domain.

Once again, one may create additional observables from the effects given in form (6) or (7) by the same arguments concerning (4).

## 3. CLASSIFICATION SCHEME FOR APPARATUSES

I have argued that the following quantum structure should generally be associated with a quantum measuring device.

*Definition 1.* Let $X$ denote the output space of a measuring apparatus. Let $\mathcal{P} = \{\Delta_i | i = 1, \ldots, n\}$ be a finite, experimentally feasible partition of $X$; i.e., $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^n \Delta_i = X$. For each $\Delta \in \mathcal{P}$, associate a positive operator $A(\Delta)$ such that $\mathrm{Tr}[\rho A(\Delta)]$ represents the relative frequency with which an event will be recorded in $\Delta$. Then all operators of the form

$$\sum_{\Delta_i \in \mathcal{P}} c_i A(\Delta_i), \qquad c_i = c(\Delta_i) \in \mathbb{C}$$

are *observables* for this apparatus.

*Definition 2.* A quantum measuring apparatus is *ideal* if there is a finest implementable finite partition of $X$ such that for each $\Delta_i$ in this finest partition, $A(\Delta_i)$ is a projection.

I remark that the meaning of the adjective "ideal" here is not to be confused with that in "ideal measurement."

*Definition 3.* A quantum measuring apparatus is *classical* if it allows arbitrarily fine partitions of outcome space $X$.

*Definition 4.* An ideal quantum measuring apparatus is *crystalline* if it has outcome space $X \subset \mathbb{R}^n$ (or an $n$-dimensional manifold) with a finest feasible partition that is a regular array of unit cells which is (locally) invariant under "lattice translations" of the form $\sum_{i=1}^{n} m_i \mathbf{e}_i$, where the $\mathbf{e}_i$ are translations along the edges of a unit cell and the $m_i$ are integers (in a bounded range for a finite crystal, but ideally exhausting the integers).

*Definition 5.* Let $\{U_g | g \in G\}$ be a unitary (antiunitary) group of symmetries for a quantum system. Then a quantum measuring apparatus is *covariant* with respect to this symmetry group if there is a natural action $g[\cdot]$ of $g$ on the outcome space $X$ such that

$$U(g)A(\Delta)U(g)^{\dagger} = A(g[\Delta]) \tag{8}$$

where $\{A(\Delta)\}$ is the set of effects associated with the apparatus and $\Delta$ is an allowed cell in any feasible partition.

*Examples.* (i) If the apparatus is crystalline (with no boundary) with $X$ = configuration space with uniform efficiency of measuring, and if $G$ is the symmetry group of this lattice, then (8) holds. (ii) In the signal analysis example, $G$ may be taken as the group of all time translations and frequency shifts. From the interpretation of time as distance from the source (distance = $ct$) and frequency shift as Doppler shift, then this $G$ becomes the Heisenberg group. (iii) If $G$ is taken as the Galilei or Poincaré group and $X$ is taken as phase space, then the action of group elements on subsets of $X$ is naturally defined and (8) represents an equivalence principle of relativity.

In what follows, we shall find that classical quantum measuring apparatuses, and in particular the covariant ones, are the most interesting to analyze. We shall also see why the adjective "classical" is appropriate.

## 4. SOME MATHEMATICAL RESULTS

I begin with some standard definitions.

*Definition 6.* Let $\Sigma$ be a *sigma algebra* on $X$; that is, $\Sigma$ is a set of subsets of $X$ such that:

(i) $X \in \Sigma$;
(ii) if $\Delta \in \Sigma$, then $X - \Delta \in \Sigma$;
(iii) if $\Delta \in \Sigma$, $i \in I$, $I$ finite or countable, then $\bigcup_{i \in I} \Delta_i \in \Sigma$.

In this case $(X, \Sigma)$ is referred to as a *measurable space*, and any subset of $X$ is *measurable* if it is in $\Sigma$. A function $f\colon X \to \mathbb{R}$ is *measurable* iff $f^{-1}((-\infty, x)) \in \Sigma$ for all $x \in \mathbb{R}$. A complex-valued function is measurable iff its real and imaginary parts are both measurable.

*Definition 7.* A function $\mu$ defined on sigma algebra $\Sigma$ is a *measure* if:

(i) $\mu: \Sigma \to \mathbb{R}_{\geq 0} \cup \{\infty\}$;

(ii) if $\{\Delta_i | i \in I\}$ is a countable family in $\Sigma$ that is disjoint ($\Delta_i \cap \Delta_j = \varnothing$ for $i \neq j$), then

$$\mu\left(\bigcup_{i \in I} \Delta_i\right) = \sum_{i \in I} \mu(\Delta_i)$$

(iii) $\mu(\Delta) < \infty$ for at least one $\Delta \in \Sigma$.

A measure $\mu$ is a (*Kolmogorov*) *probability measure* if, in addition, $\mu(X) = 1$.

A *measure space* is a triple $(X, \Sigma, \mu)$ where $(X, \Sigma)$ is a measurable space and $\mu$ is a measure on $(X, \Sigma)$.

A complex measure is a function $\mu: \Sigma \to \mathbb{C}$ satisfying (ii). Note that $|\mu(\Delta)| < \infty$ for all $\Delta \in \Sigma$ holds automatically. Any such $\mu$ can be written $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ where the $\mu_j$, $j = 1, \ldots, 4$, are (real-valued) measures.

*Definition 8.* Let $\mathscr{H}$ be a Hilbert space, $(X, \Sigma)$ a measurable space. Let $A$ be a function defined on $\Sigma$ with values in the set of positive self-adjoint operators on $\mathscr{H}$. Then $A$ is a *positive operator valued measure* (*POV* or *effect-valued measure*) if:

(i) $A(X) = 1$,

(ii) if $\{\Delta_i, i \in I\}$ is a countable, disjoint family in $\Sigma$, then $A(\bigcup_{i \in I} \Delta_i) = \sum_{i \in I} A(\Delta_i)$.

*Lemma 9.* Let $(X, \Sigma)$ be a measurable space, $\mathscr{H}$ a Hilbert space, $A$ a POV mapping $\Sigma$ to operators in $\mathscr{H}$. Let $\varphi$, $\psi \in \mathscr{H}$, and let $\rho$ be a positive trace class operator. Then $\mu_{\varphi,\varphi}(\Delta) \equiv \langle \varphi, A(\Delta)\varphi \rangle$, and $\mu_\rho(\Delta) \equiv \mathrm{Tr}[\rho A(\Delta)]$ define measures on $(X, \Sigma)$, and $\mu_{\mu,\varphi}(\Delta) \equiv \langle \psi, A(\Delta)\varphi \rangle$ defines a complex measure. If $\|\varphi\| = 1$, then $\mu_{\varphi,\varphi}$ is a probability measure. If $\rho$ has $\mathrm{Tr}\, \rho = 1$, making $\rho$ a quantum density operator, then $\mu_\rho$ is a probability measure. Note also that since $A(\Delta)$ is positive, it has a positive square root. Applying the Cauchy–Schwarz–Buniakowski inequality, we obtain

$$|\mu_{\psi,\varphi}(\Delta)|^2 = |\langle \psi, A(\Delta)\varphi \rangle|^2 \leq \mu_{\psi,\psi}(\Delta)\mu_{\varphi,\varphi}(\Delta)$$

From $A(\Delta) \leq 1$, we also obtain the general relations

$$|\mu_{\psi,\varphi}(\Delta)| \leq \|\psi\| \, \|\varphi\|$$

$$\mu_{\varphi,\varphi}(\Delta) \leq \|\varphi\|^2$$

$$\mu_\rho(\Delta) \leq \mathrm{Tr}(\rho)$$

The proofs of all these results follow immediately from the definitions and will be omitted. The proofs of the remaining theorems in this section are mostly technical, and I refer to Schroeck (1988) for them.

*Definition 10.* Let $(X, \Sigma)$ be a measurable space and let $\mathcal{H}$ be a Hilbert space. Let $A$ denote a POV from $\Sigma$ to bounded operators in $\mathcal{H}$. I then define:

$\mathscr{C} \equiv \{\sum_j c_j A(\Delta_j) | \{\Delta_j\}$ is a finite measurable partition of $X$, $c_j \in \mathbb{C}\}$.

$\mathscr{C}_u \equiv$ closure of $\mathscr{C}$ in the uniform topology.

$\mathscr{C}_{w^*} \equiv$ closure of $\mathscr{C}$ in the weak* topology, where $\mathscr{C}$ is viewed as lying in the dual of a complete base norm space, with base given by the positive trace class operators of trace one.

$\mathscr{C}_s \equiv$ closure of $\mathscr{C}$ in the following sense: let $\{A_i\}$ be a sequence in $\mathscr{C}$; let $D(B) = \{\psi \in \mathcal{H} | \{A_i\psi\}$ converges strongly$\}$; on $D(B)$ define the linear operator $B$ by $B\psi = \lim_i \{A_i\psi\}$; $\mathscr{C}_s$ is the set of all such $B$.

$\mathscr{C}_w \equiv$ closure of $\mathscr{C}$ in the following sense; let $\{A_i\}$ be a sequence in $\mathscr{C}$; let $D(B) = \{\psi \in \mathcal{H} | \langle\varphi, A_i\psi\rangle$ converges for all $\varphi$ in some dense subset of $\mathcal{H}$ which may be chosen independent of $\psi\}$; then we may define a linear operator $B$ with domain $D(B)$ by $\langle\varphi, B\psi\rangle = \lim_i \langle\varphi, A_i\psi\rangle$; $\mathscr{C}_w$ is the set of all such $B$.

One has the inclusions $\mathscr{C} \subset \mathscr{C}_u \subset \mathscr{C}_{w^*} \subset \mathscr{C}_s \subset \mathscr{C}_w$, and $\mathscr{C}_{w^*} \subset \mathscr{B}(\mathcal{H})$. Furthermore, in all cases $D(B)$ is a linear subspace of $\mathcal{H}$. It is generally not a closed subspace, and elements of $\mathscr{C}_s$, $\mathscr{C}_w$ may be unbounded operators.

For simple measurable function $s = \sum_{i \in I} c_i \chi(\Delta_i)$, $I$ finite, $c_i \in \mathbb{C}$, $\chi$ the characteristic function, define

$$\underset{\sim}{a}(s) = \sum_{i \in I} c_i A(\Delta_i) \tag{9}$$

Thus, $\underset{\sim}{a}$ provides a finite linear extension of $A$. Let us investigate limits of such operators $\underset{\sim}{a}(s)$. One trivial but useful property of $\underset{\sim}{a}$ is that for $s_1$ and $s_2$ simple measurable functions with $s_1 \leq s_2$, then, by positivity of the $A(\Delta_i)$, $\underset{\sim}{a}(s_1) \leq \underset{\sim}{a}(s_2)$.

The elements of $\mathscr{C}$ constitute the observables associated most directly with the apparatus. In the case in which there is a finest partition of the quantum apparatus which is also finite, then $\mathscr{C}$, $\mathscr{C}_u$, $\mathscr{C}_{w^*}$, $\mathscr{C}_s$, and $\mathscr{C}_w$ all coincide; so there is nothing to prove. Thus we must have a quantum measuring apparatus which is classical (in our sense) over an infinite set $X$ in order to obtain anything interesting and new. In fact, I shall be most interested in the case where $X$ is a continuum subset of $\mathbb{R}^n$. To proceed, I introduce the analog of absolute continuity but for operator-valued measures.

*Theorem 11.* Let $X = \mathbb{R}^k$, and let $\Sigma$ be the smallest sigma algebra of subsets of $X$ containing the open sets. ($\Sigma$ is called the set of Borel sets.)

Under reasonable conditions on the Hilbert space $\mathcal{H}$ and measure $\mu$ ($\mathcal{H}$ is separable and $\mu$ is a positive, sigma finite Borel measure), and if $A$ is a POV from $\Sigma$ to the bounded operators on $\mathcal{H}$ satisfying (10),

$$\text{there exists } c > 0 \text{ such that } \|A(\Delta)\| \leq c\mu(\Delta) \quad \forall \Delta \in \Sigma \tag{10}$$

then there is a family $\{\underset{\sim}{a}(x) | x \in X\}$ of positive self-adjoint operators with

$$\|\underset{\sim}{a}(x)\| \leq c \quad \text{for almost all } x \tag{11a}$$

$$A(\Delta) = \int_\Delta \underset{\sim}{a}(x)\, d\mu(x) \tag{11b}$$

Conversely, any POV satisfying (11) also satisfies (10), and $c$ may be taken to be ess $\sup_{x \in X} \|\underset{\sim}{a}(x)\|$.

I denote the condition (10) by $A \ll_c \mu$.

*Lemma 12.* Let $(X, \Sigma, \mu)$, $\mathcal{H}$, and $A$ be as above. Then the following are equivalent:

(i) $A \ll_c \mu$
(ii) $\mu_{\psi,\varphi}(\Delta) \leq c\|\psi\|\,\|\varphi\|\mu(\Delta)$ for all $\psi, \varphi \in \mathcal{H}$
(iii) $\mu_{\psi,\psi}(\Delta) \leq c\|\psi\|^2\mu(\Delta)$ for all $\psi \in \mathcal{H}$
(iv) $\mu_\rho(\Delta) \leq c \operatorname{Tr} \rho\, \mu(\Delta)$ for all positive trace class operators $\rho$
(v) $\mu_{\psi,\varphi}(\Delta) \leq c\mu(\Delta)$ for all $\psi, \varphi \in \mathcal{H}$, $\|\psi\| = \|\varphi\| = 1$
(vi) $\mu_{\psi,\psi}(\Delta) \leq c\mu(\Delta)$ for all $\psi \in \mathcal{H}$, $\|\psi\| = 1$
(vii) $\mu_\rho(\Delta) \leq c\mu(\Delta)$ for all density operators $\rho$.

I only mention this result since these various definitions of "absolute continuity" may occur in the literature, each being in some sense a natural definition. For example, see Holevo (1982, p. 52).

*Theorem 13.* Let $(X, \Sigma, \mu)$ be a measure space, $\mathcal{H}$ a Hilbert space, $A$ a POV from $\Sigma$ to bounded operators on $\mathcal{H}$ satisfying $A \ll_c \mu$. Then (9) extends uniquely to a linear map from $L^p(X, \mu)$, $1 \leq p \leq \infty$, to the set of bounded operators in $\mathcal{H}$ satisfying, for $f$ real,

$$\|\underset{\sim}{a}(f)\| \leq c^{1/p}\|f\|_p$$

If $f$ is complex-valued, then dividing into the real and imaginary parts, the inequality above needs only be modified by a factor of 2. [Recall $L^p(X, \mu) = \{f: X \to \mathbb{C} \text{ such that } \int |f(x)|^p\, d\mu(x) < \infty\}$, $1 \leq p < \infty$. Then define $\|f\|_p = [\int |f(x)|^p\, d\mu(x)]^{1/p}$; $\|f\|_\infty = \text{ess sup}|f(x)|$.] Furthermore, one has the representation

$$\underset{\sim}{a}(f) = \int_X f(x)\underset{\sim}{a}(x)\, d\mu(x) \tag{12}$$

where $\underset{\sim}{a}(x)$ is the density for $A$ obtained in Theorem 11. In case $f$ is also real-valued, then $\underset{\sim}{a}(f)$ is in fact a self-adjoint operator.

*Definition 14.* Let $A$ be a compact operator in $\mathscr{H}$; i.e., $(AA^{\dagger})^{1/2}$ has purely discrete spectrum $\{\alpha_k\}$ with each $\alpha_k$ of finite multiplicity, except for perhaps an accumulation point at 0. Then $A$ is of trace class $\mathscr{B}_p$ iff $\sum_k (\alpha_k)^p < \infty$.

*Theorem 15.* (Schroeck, 1988). Under the conditions of Theorem 13 and with the additional condition $\mathrm{Tr}[\underset{\sim}{a}(x)] \leq K$ for almost all $x \in X$, with $K$ a fixed constant, and for $f \in L^p(X, \mu)$, where $1 \leq p \leq \infty$, then $\underset{\sim}{a}(f) \in \mathscr{B}_p$. Since these classes are all classes of compact operators, it follows that each such $\underset{\sim}{a}(f)$ for $f$ real-valued has purely discrete spectrum and a complete orthogonal set of eigenvectors. For the case $p = 1$, one furthermore has $\mathrm{Tr}[A(f)] \leq K \|f\|_1$, and if $f$ is nonnegative, this is even equality.

I should add a remark here that this result is somewhat surprising, since these $\underset{\sim}{a}(f)$ are obtained by making generally unsharp measurements to get a POV in the first place, and then further smearing out the data by integrating with respect to $f\,d\mu$. For example, if the $f$ describes a confidence function, then $f \in L^1(X, \mu)$; so $\underset{\sim}{a}(f)$ is trace class and hence has purely discrete spectrum, etc. The special choice $f(x) = [\mu(\Delta)]^{-1}\chi_\Delta(x)$ is also $L^1(X, \mu)$ for $\Delta \in \Sigma$, $\mu(\Delta) < \infty$, and in this case $\underset{\sim}{a}(f) = A(\Delta)$, a simple effect. Hence, the effects in this situation have eigenvectors, and even a complete orthogonal set of them. However, whenever $f$ is a confidence function, then the spectrum of $\underset{\sim}{a}(f)$ is strictly less than 1 (and nonnegative), so that the system always attenuates its eigenmodes. "Measurement with probability one" is excluded. This answers a question raised some years past (Schroeck, 1981).

Of course, all of this depends on the density being trace class. This is not an unusual circumstance. It occurred in our signal analysis problem in which each $\underset{\sim}{a}(x)$ was a one-dimensional projection, hence having trace $= 1$. Neither is it unusual in the case of localization observables which are covariant with respect to the Galilei or Poincaré group, as the following two theorems show.

*Theorem 16.* Let $(X, \Sigma, \psi)$, $A$, and $\mathscr{H}$ be as before, and $A \ll_c \mu$ with density $\underset{\sim}{a}(x)$. Let $A$ satisfy the covariance condition (8) with respect to continuous symmetry group $G$. Then $\underset{\sim}{a}(x)$ is a continuous function of $x$, and

$$U(g)\underset{\sim}{a}(x)U(g)^{\dagger} = \underset{\sim}{a}(g[x]) \tag{13}$$

Consequently, $\|a(g[x])\| = \|a(x)\|$ and $\mathrm{Tr}(\underset{\sim}{a}(g[x])) = \mathrm{Tr}(\underset{\sim}{a}(x))$; so we only need a bound on (the trace of) $\underset{\sim}{a}$ at one point $x \in X$ in order to get a bound at all points in the orbit $\{g[x] | g \in G\}$ of $x$. If $G$ is transitive on $X$, i.e., if this orbit is all of $X$, then we may take $c = \|\underset{\sim}{a}(x)\|$ at essentially any $x$.

*Theorem 17.* For the covariant localization operators obtained in the phase space representations of the Galilei and Poincaré groups (Ali and Prugovečki 1986; Ali *et al.*, 1988; Brooke and Schroeck, to appear), all conditions of Theorem 16 are met and $\underset{\sim}{a}(x)$ is either a density operator (trace = 1) or, in the case of irreducible representations, a one-dimensional projection (trace = 1). Hence, we may also take $c = 1$ in these cases. Furthermore, the $\underset{\sim}{a}(x)$ are strongly continuous functions of $x$.

If we couple this result with Theorem 15, we obtain $K = 1$, and $\mathrm{Tr}[A(f)] = \|f\|_1$ for $f \in L^1(X, \mu)$, $f$ nonnegative. This is a form of uncertainty relation closely connected with the channel capacity theorem in signal analysis. To see this connection, let $\Delta$ be a rectangle in phase space of dimension $2d$ (or in the time-frequency domain) of width $\Delta Q$ and height $\Delta P$ (or time width $2T$ and bandwidth $W$). Let $f$ be the characteristic function of $\Delta$. Then $\mu(\Delta) = (2\pi\hbar)^{-d} \Delta Q \Delta P$, and similarly in the time-frequency case. Next let $\{\alpha_k\}$ denote the set of eigenvalues listed in decreasing order and listed as often as the multiplicity. Now, $\sum_k \alpha_k = (2\pi\hbar)^{-d} \Delta Q \Delta P$. Let us choose $B \in (0, 1)$ to represent the lowest attenuation with which we agree that a wavefunction may be considered to be localized in $\Delta$; that is, $\psi$ is considered localized in $\Delta$ iff $\|A(\Delta)\psi\| \geq B\|\psi\|$. Suppose that $N$ = the number of the $\alpha_k$ counted with multiplicity with $\alpha_k \geq B$. Then $NB \leq \mathrm{Tr}[A(f)] = (2\pi\hbar)^{-d} \Delta Q \Delta P$. The spectrum of the localization operators $A(\Delta)$ in the solved cases all have $\alpha_k$ initially close to 1, and remaining close to one until they drop precipitously to close to zero (Slepian and Polak, 1961; Landau and Polak, 1961, 1962; Slepian, 1964; Daubechies, 1988). Thus, we may effectively take $B = 1$ to get a bound on $N$. In signal processing this is called the 2TW theorem, or the channel capacity theorem. [For further applications of the theory to signal processing and experimental physics see Healy and Schoeck (1988).] I also remark that if $\Delta P \Delta Q < (2\pi\hbar)^d$, then $NB < 1$; so effectively no state may be localized with this much precision without severe attenuation.

There are of course more general cases than $f \in L^p(X, \mu)$. For example, no polynomial is of any $L^p$ type. To handle these cases, we return to (12) and consider (12) to be defined weakly on pairs of vectors $\psi, \varphi \in \mathcal{H}$ for which

$$\int f(x)\, d\mu_{\psi,\varphi}(x) = \left\langle \psi, \int f(x) \underset{\sim}{a}(x)\, d\mu(x) \varphi \right\rangle \tag{14}$$

exists. For fixed $f$, the set of such $\psi$ and the set of such $\varphi$ each forms a linear space. In this way we may obtain $\int f(x) \underset{\sim}{a}(x)\, d\mu(x)$ as the operator density for a bilinear form. Furthermore, the left-hand side of (14), being a standard integral of a measurable function $f$, may be obtained as a limit of integrals of simple measurable functions $s_n$ with $s_n \to f$. Thus, once again, the relation (9) completely determines all functions defined through (14) and leads us from $\mathscr{C}$ to $\mathscr{C}_{w}$. It also takes the range out of the set of bounded operators, and out of the set of operators with purely discrete spectrum, as the following example shows.

Let $G$ = Heisenberg group, $X = \Gamma$ = phase space with points denoted $(q, p)$, $q$ = position, $p$ = momentum, $\underset{\sim}{a}(q, p)$ = projection onto the basic harmonic oscillator state with expected position = $q$, expected momentum = $p$, $d\mu(q, p) = (2\pi\hbar)^{-1}\, dq\, dp$. Then (Schroeck, 1981) for $f(q, p) = q$ we have $\underset{\sim}{a}(f) = Q$ = position operator, and for $f(q, p) = p$ we obtain $\underset{\sim}{a}(f) = P$ = momentum operator. Hence these coexistent observables are unbounded self-adjoint operators, with purely continuous spectrum and in fact observables which do not commute.

For another example of noncommuting observables which are coexistent (albeit bounded) see Schroeck (1982), in which a model is given in which different components of spin are coexistent.

Finally, if the $\underset{\sim}{a}(x)$ are one-dimensional projections onto a subspace spanned by vector $\psi$, and $\psi$ is such that $\langle\psi, U(g)\psi\rangle \neq 0$ for almost all $g \in G$, then the set $\{a(x)\}$ is informationally complete (Healy and Schroeck, 1988). From a recent theorem (Busch, 1988), one then knows that $\mathscr{C}_{w^*}$ is the set of all bounded operators in $\mathscr{H}$.

## 5. CONNECTION WITH QUANTIZATION

Let us assume from the start that we have a POV $A$ with density $\underset{\sim}{a}$, trace of $\underset{\sim}{a}(x) = 1$, $A$ covariant with respect to symmetry group $G$ of the usual types of symmetries: translations, rotations, and other symmetries of the Galilei group. We shall also take $X = \Gamma$ = phase space of dimension $2d$. Let $\rho$ be any density operator. Then $\mu_\rho$ is a classical probability distribution with continuous density $\mu'_\rho(q, p) = \mathrm{Tr}[\rho\underset{\sim}{a}(q, p)]$. Notice now that the classical expectation of the classical observable (measurable function) $f$ is given by

$$\begin{aligned}
\text{classical expected value} &= \int f(q, p)\, d\mu_\rho(q, p) \\
&= \int f(q, p)\mu'_\rho(q, p)(2\pi\hbar)^{-d}\, dq\, dp \\
&= \int f(q, p)\, \mathrm{Tr}[\rho\underset{\sim}{a}(q, p)](2\pi\hbar)^{-d}\, dq\, dp \\
&= \mathrm{Tr}[\rho\underset{\sim}{a}(f)] \\
&= \text{quantum expected value.}
\end{aligned}$$

Thus the pair of mappings

(i) $\rho \to \mu'_\rho(q, p)$, (ii) $f \to \underset{\sim}{a}(f)$

provide, respectively, (i) a dequantization of states and (ii) a quantization of observables, which preserves expectation values. It also establishes the connection Poisson bracket $\leftrightarrow$ commutator [Lie] bracket and connects quantum and classical dynamics (Schroeck, 1985*c*). In fact either of (i), (ii) implies the other (Guz, 1984*a,b*). Notice that this quantization/dequantization scheme depends essentially on the density $\underset{\sim}{a}(x)$ and hence essentially on the POV $A$, which in turn is determined by the measuring apparatus. In the choice $\underset{\sim}{a}(q, p)$ = ground state of the harmonic oscillator described above, then, the map $f \to \underset{\sim}{a}(f)$ is antinormal ordering, as I shall show below. For a general $\underset{\sim}{a}(x)$, one could compute directly which operators the choices $q$, $p$, $qp$, $pq$, $qp^2$, etc., for $f$ yield. [The $\underset{\sim}{a}(f)$ are functions of the $Q$ and $P$ operators. Thus, the apparatus "solves" the ordering problem in that it completely determines this map $f \to \underset{\sim}{a}(f)$, and the ordering is completely determined by the map $f \to \underset{\sim}{a}(f)$.]

Another point concerning this quantization/dequantization scheme is that it establishes an *equivalence* between the quantum formalism and the classical one without taking the limit $\hbar \to 0$. Since $\hbar$ is a physical *constant*, this is a good thing.

A further point is that once we have a "classical" quantum measuring apparatus, we obtain this equivalence between the classical and quantum representations of the system; that is, such a quantum system can be accurately described classically!

To see that the choice $\underset{\sim}{a}(q, p)$ = projection onto the ground state of the harmonic oscillator as above gives normal ordering, recall that the harmonic oscillator ground-state wave function is annihilated by the lowering operator. I will demonstrate with a $d = 1$ calculation. After shifting in phase space by $(q, p)$, this annihilation property reads

$$(P + iQ)\underset{\sim}{a}(q, p) = (p + iq)\underset{\sim}{a}(p, q) \tag{15}$$

and hence

$$(P + iQ)^n \underset{\sim}{a}(p, q) = (p + iq)^n \underset{\sim}{a}(p, q)$$

Taking adjoints, we obtain

$$\underset{\sim}{a}(p, q)(P - iQ)^n = (p - iq)^n \underset{\sim}{a}(p, q)$$

$$(P + iQ)^m \underset{\sim}{a}(p, q)(P - iQ)^n = (p + iq)^m (p - iq)^n \underset{\sim}{a}(p, q)$$

From (2c), (12), and this we obtain

$$(2\pi\hbar)^{-1} \int (p - iq)^m (p + iq)^n \underset{\sim}{a}(p, q)\, dq\, dp = (P + iQ)^n (P - iQ)^m \tag{16}$$

Using (16) and $p=[(p+iq)+(p-iq)]/2$, and similarly for $q$, one can now easily compute the quantized version of $p^m q^n$ or of any polynomial in $p$ and $q$, thereby showing that this quantization corresponds to normal ordering. The $d=3$ calculations are almost identical.

## 6. CONCLUSION

We have seen that for large classes of quantum measuring apparatuses, the apparatus itself determines a POV, the set of all coexistent observables that can be measured with the apparatus, and a quantization/dequantization scheme without the $\hbar \to 0$ limit. This connection allows a new interpretation of the concept of a classical apparatus for measuring a quantum system, circumventing a semiclassical treatment. The class of coexistent observables so obtained is generally not commutative. This class may be an informationally complete set of observables, which is a vast improvement over the spectral measures (projection-valued measures) of position only and of momentum only, even when combined. We obtain new ways of expressing uncertainty relations, applying them straightforwardly to signal analysis and other areas of experimental physics. The analysis here extends the class of coexistent observables beyond the set of simple effects determined by the apparatus and gives a foundation for the physical interpretation of this analog of the spectral theorem for integration with respect to a POV. Finally, each such "classical" apparatus determines its own quantization/dequantization scheme, its own solution to the ordering problem, and its own classical representation of the quantum system.

## REFERENCES


Ali, S. T., and Emch, G. G. (1974). *Journal of Mathematical Physics*, **15**, 176–182.

Ali, S. T., and Prugovečki, E. (1986). *Acta Applicandae Mathimaticae*, **6**, 1–62.

Ali, S. T., Brooke, J., Busch, P., Gagnon, R., and Schroeck, F. E., Jr. (1988). *Canadian Journal of Physics*, **66**, 238–244.

Brooke, J., and Schroeck, F. E., Jr. (to appear). Half integer spin representations of the Poincaré group on phase space, in *Proceedings of the International Symposium on Spacetime Symmetries*, Y. S. Kim and W. W. Zachary, eds. (*Nuclear Physics B*, to appear).

Busch, P. (1985*a*). Can quantum theoretical reality be considered sharp?, in *Recent Developments in Quantum Logic*, P. Mittelstaedt and E. W. Stachow, eds., Mannheim Bibliographisches Institut, pp. 81–101.

Busch, P. (1985*b*). Elements of unsharp reality in the E.P.R. experiment, in *Symposium on the Foundations of Modern Physics*, P. Lahti and P. Mittelstaedt, eds., World Scientific, Singapore, pp. 343–357.

Busch, P. (1988). A characterization of informationally complete sets of physical quantities, University of Cologne, Preprint.

Busch, P., and Lahti, P. J. (1989). The determinatoin of the past and the future of a physical system in quantum mechanics, *Foundations of Physics*, to appear.

Corbett J. V., and Hurst, C. A. (1978). Are wave functions uniquely determined by their position and momentum distributions?, *Journal of the Australian Mathematical Society* **20** (Series B), 182-201.

Daubechies, I. (1988). The wavelet transform, time-frequency localization and signal analysis, AT&T Bell Labs, Preprint.

Davies, E. B. (1970). *Journal of Functional Analysis*, **6**, 318-346.

Davies, E. B., and Lewis, J. T. (1970). *Communications in Mathematical Physics*, **17**, 239-260.

Emch, G. G. (1972). *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Wiley, New York, pp. 33-42.

Guz, W. (1984*a*). *International Journal of Theoretical Physics*, **23**, 157-184.

Guz, W. (1984*b*). *Foundations of Physics*, **12**, 821-848.

Healy, D. M., and Schroeck, F. E., Jr. (1988). Application of stochastic quantum mechanics and coherent state methodologies to signal processing, Preprint.

Holevo, A. S. (1982). *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, New York.

Jauch, J. M., and Piron, C. (1967). *Helvetica Physica Acta*, **18**, 559-570.

Klauder, J. (1964). *Journal of Mathematical Physics*, **5**, 177-187; n.b. Theorem 2.

Lahti, P. J. (1985). In *Developments in Quantum Logic*, P. Mittelstaedt and E. W. Stachow, eds., Mannheim Bibliographisches Institut, pp. 61-80.

Landau, H. J., and Polak, H. O. (1961). *Bell System Technical Journal*, **1961**, 65-84.

Landau H. J., and Polak, H. O. (1962). *Bell System Technical Journal*, **1962**, 1295-1336.

Ludwig, G. (1985). *An Axiomatic Basis for Quantum Mechanics*, Springer-Verlag, New York.

Ozawa, M., (1984). *Journal of Mathematical Physics*, **25**, 79-87.

Pavčič, M. (1985). When do position and momentum distributions determine the quantum mechanical state?, *Physics Letters A*, **118**, 5-7.

Progovečki, E. (1984). *Stochastic Quantum Mechanics and Quantum Spacetime*, D. Reidel, Dordrecht, pp. 139-143.

Schroeck, F. E., Jr. (1981). *Journal of Mathematical Physics*, **22**, 2562-2572.

Schroeck, F. E., Jr. (1982). *Foundations of Physics*, **12**, 479-497.

Schroeck, F. E., Jr. (1985*a*). Entropy and measurement in stochastic quantum mechanics, in *Symposium on the Foundations of Modern Physics*, P. Lahti and P. Mittelstaedt, eds., World Scientific, Singapore, pp. 573-590.

Schroeck, F. E., Jr. (1985*b*). *Foundations of Physics*, **15**, 677-681.

Schroeck, F. E., Jr. (1985*c*). *Journal of Mathematical Physics*, **26**, 306-310.

Schroeck, F. E., Jr. (1988). Integration of complex-valued functions with respect to a positive operator valued measure, Florida Atlantic University, Preprint.

Slepian, D. (1964). *Bell System Technical Journal*, **1964**, 3009-3057.

Slepian, D., and Polak, H. O. (1961). *Bell System Technical Journal*, **1961**, 43-63.

Vogt, A., (1978). Position and momentum distributions do not determine the quantum mechanical state, in *Mathematical Foundations of Quantum Theory*, A. R. Marlow, ed., Academic Press, New York, pp. 365-372.

Von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey, pp. 214-223.